# MTH 605: Topology I

## Practice Assignment VII

1. Show that any covering space of the torus $T$ is isomorphic to $T$, or the plane, or $S^1 \times \mathbb{R}$.

2. Show that for $g \geq 1$, the closed nonorientable surface $N_g$ with $g$ crosscaps is double-covered by the closed orientable surface $S_{g-1}$. Derive the subgroup of $\pi_1(N_g)$ that corresponds to this cover.

3. Consider the closed orientable surface $S_g$ of genus $g \geq 2$.

   (a) Show that any $k$-sheeted cover of $S_g$ is homeomorphic to $S_{(g-1)k+1}$.

   (b) Consider the the covering space $p_k : S_{(g-1)k+1} \to S_g$ induced by the covering action of $\mathbb{Z}_k$ on $S_{(g-1)k+1}$. Derive the subgroup of $\pi_1(S_g)$ that corresponds to this cover.

4. For $g \geq 0$ and $b \geq 1$, let $S_{g,b}$ be the orientable surface of genus $g$ with $b$ boundary components (i.e. $S_{g,b}$ is homeomorphic to the surface obtained by the deleting $b$ disjoint open disks from $S_g$.)

   (a) Describe a cell-complex structure on $S_{g,b}$.

   (b) Use this structure to derive a presentation for $\pi_1(S_{g,b})$.

5. Find all connected 2-sheeted, 3-sheeted, and 4-sheeted covering spaces of the figure 8 space $S^1 \vee S^1$ up to isomorphism.

6. Assume that $\pi_1(S^1 \vee S^1) \cong \langle a, b \rangle$, the free group on two letters $a$ and $b$. Up to isomorphism, find the covering spaces of $S^1 \vee S^1$ that corresponds to the subgroups $\langle a \rangle \leq \pi_1(S^1 \vee S^1)$ and $\langle b \rangle \leq \pi_1(S^1 \vee S^1)$.

7. Consider the covering space $p : \tilde{X} \to X$ of the wedge $X$ of three circles shown in the figure below.

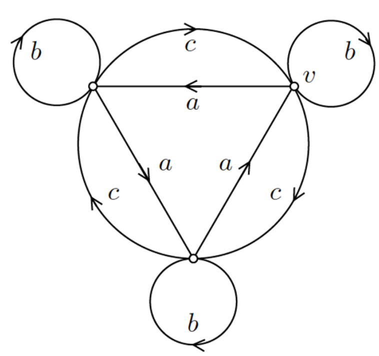


Figure 1: A cover $\tilde{X}$ of the wedge of three circles.

   (a) Show that $\tilde{X}$ is a normal cover. [5]

(b) Find a free generating set for the subgroup $p_*(\pi_1(\tilde{X}, v))$ of $\pi(X) = \langle a, b, c\rangle$ that corresponds to this cover. [5]

8. Given a group and a normal subgroup $N$, show that there exists a normal covering $\widetilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\widetilde{X}) \cong N$. and the deck transformation group $G(\widetilde{X} \to X) \approx G/N$.

9. Given covering actions of groups $G_1$ on $X_1$ and $G_2$ on $X_2$, show that the action of $G_1 \times G_2$ on $X_1 \times X_2$ defined by $(g_1, g_2)(x_1, x_2) = (g_1(x_1), g_2(x_2))$ is a covering space action, and that $(X_1 \times X_2)/(G_1 \times G_2) \approx X_1/G_1 \times X_2/G_2$.

10. For a covering space $p : \widetilde{X} \to X$ with $X$ connected, locally path-connected, and semilocally simply-connected, show that

    (a) The components of $\widetilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

    (b) Under the Galois correspondence between connected covering spaces of $X$ and the subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\widetilde{X}$ containing a given lift $\widetilde{x_0}$ of $x_0$ is the stabilizer of $\widetilde{x_0}$, the subgroup consisting of elements whose action on the fiber leaves $\widetilde{x_0}$ fixed.

11. Let $X$ and $Y$ be topological spaces. Suppose there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq i_Y$ and $g \circ f \simeq i_X$. Then $X$ is said to be *homotopically equivalent* to $Y$ (or $X$ is said to have the *same homotopy type* as $Y$), denoted by $X \simeq Y$.

    (a) Please read Lemma 58.4, Corollaries 58.5-58.6, and Theorem 58.7 from Munkres.

    (b) Show that if a space $X$ is contractible, then $X$ has the same homotopy type as a point.

    (c) Show that if $A$ is a deformation retract of $X$.

    (d) Give a counterexample to show that the converse of (b) does not hold always. [Hint: Consider $X = \cup_{n \in \mathbb{Z}_+} (1/n) \times I$ and $x_0 = (0, 1)$.]